

# A boundary value problem for the five-dimensional stationary rotating black holes

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## Abstract

We study the boundary value problem for the stationary rotating black hole solutions to the five-dimensional vacuum Einstein equation. Assuming the two commuting rotational symmetry and the sphericity of the horizon topology, we show that the black hole is uniquely characterized by the mass, and a pair of the angular momenta.

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## I. INTRODUCTION

In recent years there has been renewed interest in higher dimensional black holes in the context of both string theory and brane world scenario. In particular, the possibility of black hole production in linear collider is suggested [1, 2, 3, 4]. Such phenomena play a key role to get insight into the structure of space-time; we might be able to prove the existence of the extra dimensions and have some information about the quantum gravity. Since the primary signature of the black hole production in the collider will be Hawking emission from the stationary black hole, the classical equilibrium problem of black holes is an important subject. The black holes produced in colliders will be small enough compared with the size of the extra dimensions and generically have angular momenta, they will be well approximated by higher dimensional rotating black hole solutions found by Myers and Perry [5]. The Myers-Perry black hole which has the event horizon with spherical topology can be regarded as the higher-dimensional generalization of the Kerr black hole. One might expect that such a black hole solution describes the classical equilibrium state continued from the black hole production event, if it equips stability and uniqueness like the Kerr black hole in four-dimensions. The purpose of this paper is to consider the uniqueness and nonuniqueness of the rotating black holes in higher dimensions.

The uniqueness theorem states that a four-dimensional black hole with regular event horizon is characterized only by mass, angular momentum and electric charge [6, 7]. Recently, uniqueness and nonuniqueness properties of five or higher-dimensional black holes are also studied. Emparan and Reall have found a black ring solution of the five-dimensional vacuum Einstein equation, which describes a stationary rotating black hole with the event horizon homeomorphic to  $S^2 \times S^1$  [8]. In a certain parameter region, a black ring and a (Myers-Perry) black hole can carry the same mass and angular momentum. This might suggest the nonuniqueness of higher-dimensional stationary black hole solutions. For example, Reall [9] conjectured the existence of stationary, asymptotically flat higher-dimensional vacuum black hole admitting exactly two commuting Killing vector fields although all known higher dimensional black hole solutions have three or more Killing vector fields. In six or higher dimensions, Myers-Perry black hole can have an arbitrarily large angular momentum for a fixed mass. The horizon of such black hole highly spreads out in the plane of rotation and looks like a black brane in the limit where the angular momentum goes to infinity. Hence,

Empanan and Myers [10] argued that rapidly rotating black holes are unstable due to the Gregory-Laflamme instability [11] and decay to the stationary black holes with rippled horizons implying the existence of black holes with less geometric symmetry compared with the Myers-Perry black holes. For supersymmetric black holes and black rings, string theoretical interpretation are given by Elvang and Emparan [12]. They showed that the black hole and the black ring with same asymptotic charges correspond to the different configurations of branes, giving a partial resolution of the nonuniqueness of supersymmetric black holes in five dimensions. On the other hand, we have uniqueness theorems for black holes at least in the static case [13, 14, 15, 16, 17, 18]. Furthermore, the uniqueness of the stationary black holes is supported by the argument based on linear perturbation of higher dimensional static black holes [19, 20]. There exist regular stationary perturbations that fall off at asymptotic region only for vector perturbation, and then the number of the independent modes corresponds to the rank of the rotation group, namely the number of angular momenta carried by the Myers-Perry black holes [21]. This suggests that the higher-dimensional stationary black holes have uniqueness property in some sense, but some amendments will be required. Here we consider the possibility of restricted black hole uniqueness which is consistent with any argument about uniqueness or nonuniqueness. Though the existence of the black ring solution explicitly violates the black hole uniqueness, there still be a possibility of black hole uniqueness for fixed horizon topology [22]. Hence we restrict ourselves to the stationary black holes with spherical topology.

In this paper, we consider the asymptotically flat, black hole solution to the five-dimensional vacuum Einstein equation with the regular event horizon homeomorphic to  $S^3$ , admitting two commuting spacelike Killing vector fields and stationary (timelike) Killing vector field. The two spacelike Killing vector fields correspond to the rotations in the  $(X^1-X^2)$ -plane and  $(X^3-X^4)$ -plane in the asymptotic region ( $\{X^\mu\}$  are the asymptotic Cartesian coordinates), respectively, which are commuting with each other. Along with the argument by Carter [23], it is possible to construct a timelike Killing vector field tangent to the fixed points (namely, axis) of the axi-symmetric Killing vector field from the given timelike Killing vector field. Repeating this procedure for each commuting spacelike Killing vector field, the obtained timelike Killing vector field is also commuting with both spacelike Killing vector fields. Hence, it is natural to assume all the three Killing vector fields are commuting with each other. The five-dimensional vacuum space-time admitting three commuting Killing

vector fields is described by the nonlinear  $\sigma$ -model [24]. Then the Mazur identity [25] for this system is derived. We show that the five-dimensional black hole solution with regular event horizon of spherical topology is determined by three parameters under the appropriate boundary conditions.

The remainder of the paper is organized as follows. In Section II A, we give the field equations for the five-dimensional vacuum space-time admitting three commuting Killing vector fields. In Section II B, we introduce the matrix form of field equations to clarify the hidden symmetry of this system following Maison [24]. Then the Mazur identity which is useful to show the coincidence of two solutions is derived in Section III. In Section IV, we determine the boundary conditions. We summarize this paper and make discussions on related matters in Section V.

## II. FIVE-DIMENSIONAL VACUUM SPACE-TIME ADMITTING THREE COMMUTING KILLING VECTOR FIELDS

Assuming the symmetry of space-time, the Einstein equations reduce to the equations for the scalar fields defined on three-dimensional space. Then, we show that the system of the scalar fields is described by a nonlinear  $\sigma$ -model.

### A. Weyl-Papapetrou metrics

We consider the five-dimensional space-time admitting two commuting Killing vector fields  $\xi_I = \partial_I$ , ( $I = 4, 5$ ). The metric can be written in the form

$$g = f^{-1}\gamma_{ij}dx^i dx^j + f_{IJ}(dx^I + w_i^I dx^i)(dx^J + w_j^J dx^j), \quad (1)$$

where  $i, j = 1, 2, 3$ ,  $f = \det(f_{IJ})$ . The three-dimensional metric  $\gamma_{ij}$ , the functions  $w_i^I$  and  $f_{IJ}$  are independent on the coordinates  $x^I$  ( $x^4 = \phi$ ,  $x^5 = \psi$ , and we will later identify  $\xi_4$  and  $\xi_5$  as Killing vector fields corresponding to two independent rotations in the case of asymptotically flat space-time). We define the twist potential  $\omega_I$  by

$$\omega_{I,\mu} = f f_{IJ} \sqrt{|\gamma|} \epsilon_{ij\mu} \gamma^{im} \gamma^{jn} \partial_m w_n^J, \quad (2)$$

where  $\mu = 1, \dots, 5$ ,  $\gamma = \det(\gamma_{ij})$ ,  $\gamma^{ij}$  is the inverse metric of  $\gamma_{ij}$ , and  $\epsilon_{\lambda\mu\nu}$  denotes the totally skew-symmetric symbol such that  $\epsilon_{123} = 1, \epsilon_{I\mu\nu} = 0$ . Then the vacuum Einstein

equation reduces to the field equations for the five scalar fields  $f_{IJ}$  and  $\omega_I$  defined on the three-dimensional space:

$$D^2 f_{IJ} = f^{KL} Df_{IK} \cdot Df_{JL} - f^{-1} D\omega_I \cdot D\omega_J, \quad (3)$$

$$D^2 \omega_I = f^{-1} Df \cdot D\omega_I + f^{JK} Df_{IJ} \cdot D\omega_K, \quad (4)$$

and the Einstein equations on the three-dimensional space:

$${}^{(\gamma)}R_{ij} = \frac{1}{4} f^{-2} f_{,i} f_{,j} + \frac{1}{4} f^{IJ} f^{KL} f_{IK,i} f_{JL,j} + \frac{1}{2} f^{-1} f^{IJ} \omega_{Ii} \omega_{Jj}, \quad (5)$$

where  $D$  is the covariant derivative with respect to the three-metric  $\gamma_{ij}$  and the dot denotes the inner product determined by  $\gamma_{ij}$ .

Here we assume the existence of another Killing vector field  $\xi_3 = \partial_3$  which commutes with the other Killing vectors as  $[\xi_3, \xi_I] = 0$  (we will later identify the  $\xi_3$  as the stationary Killing vector field in the case of asymptotically flat space-time). Then the metric can be written in the Weyl-Papapetrou-type form [26]

$$g = f^{-1} e^{2\sigma} (d\rho^2 + dz^2) - f^{-1} \rho^2 dt^2 + f_{IJ} (dx^I + w^I dt)(dx^J + w^J dt), \quad (6)$$

where we denote  $x^3 = t$ , and all the metric functions depend only on  $\rho$  and  $z$ . Once the five scalar fields  $f_{IJ}, \omega_I$  are determined, the other metric functions  $\sigma$  and  $w^I$  are obtained by solving the following partial derivative equations:

$$\begin{aligned} \frac{2}{\rho} \sigma_{,\rho} &= \frac{1}{4} f^{-2} [(f_{,\rho})^2 - (f_{,z})^2] + \frac{1}{4} f^{IJ} f^{MN} (f_{IM,\rho} f_{JN,\rho} - f_{IM,z} f_{JN,z}) \\ &\quad + \frac{1}{2} f^{-1} f^{IJ} (\omega_{I,\rho} \omega_{J,\rho} - \omega_{I,z} \omega_{J,z}), \end{aligned} \quad (7)$$

$$\frac{1}{\rho} \sigma_{,z} = \frac{1}{4} f^{-2} f_{,\rho} f_{,z} + \frac{1}{4} f^{IJ} f^{MN} f_{IM,\rho} f_{JN,z} + \frac{1}{2} f^{-1} f^{IJ} \omega_{I,\rho} \omega_{J,z}, \quad (8)$$

$$w_{,\rho}^I = \rho f^{-1} f^{IJ} \omega_{J,z}, \quad (9)$$

$$w_{,z}^I = -\rho f^{-1} f^{IJ} \omega_{J,\rho}. \quad (10)$$

The  $f_{IJ}$  and  $\omega_I$  are given by axi-symmetric solution of the field equations (3) and (4) on the abstract flat three-space with the metric

$$\gamma = d\rho^2 + dz^2 + \rho^2 d\varphi^2. \quad (11)$$

Thus the system is described by the action

$$S = \int d\rho dz \rho \left[ \frac{1}{4} f^{-2} (\partial f)^2 + \frac{1}{4} f^{IJ} f^{KL} \partial f_{IK} \cdot \partial f_{JL} + \frac{1}{2} f^{-1} f^{IJ} \partial \omega_I \cdot \partial \omega_J \right]. \quad (12)$$

## B. Matrix representation

The action (12) is invariant under the global  $SL(3, \mathbf{R})$  transformations as shown by Maison [24]. Instead of the nonlinear representation by the scalar fields  $f_{IJ}$  and  $\omega_I$ , we introduce the  $SL(3, \mathbf{R})$  matrix field  $\Phi$  as

$$\Phi = \begin{pmatrix} f^{-1} & -f^{-1}\omega_\phi & -f^{-1}\omega_\psi \\ -f^{-1}\omega_\phi & f_{\phi\phi} + f^{-1}\omega_\phi\omega_\phi & f_{\phi\psi} + f^{-1}\omega_\phi\omega_\psi \\ -f^{-1}\omega_\psi & f_{\phi\psi} + f^{-1}\omega_\phi\omega_\psi & f_{\psi\psi} + f^{-1}\omega_\psi\omega_\psi \end{pmatrix}, \quad (13)$$

which is symmetric ( ${}^t\Phi = \Phi$ ) and unimodular ( $\det \Phi = 1$ ).  $\Phi$  transforms as a covariant, symmetric, second-rank tensor fields under global  $SL(3, \mathbf{R})$  transformations. When the Killing vector fields  $\xi_\phi$  and  $\xi_\psi$  are spacelike, all the eigenvalues of  $\Phi$  are real and positive. Therefore, there is an  $SL(3, \mathbf{R})$  matrix field  $g$  which is a square root of the matrix field  $\Phi$ , namely

$$\Phi = g {}^t g. \quad (14)$$

This square root matrix  $g$  is determined upto global  $SO(3)$  rotation because the rotation  $g \mapsto g\Lambda$  for any  $\Lambda \in SO(3)$  is canceled by  $\Lambda^{-1} = {}^t\Lambda$ . Since any  $SL(3, \mathbf{R})$  matrix field  $g$  conversely defines a symmetric and unimodular matrix field by  $\Phi = g {}^t g$ , the matrix  $\Phi$  defines a map from two-dimensional  $\rho$ - $z$ -half plane (base space) to the coset space  $SL(3, \mathbf{R})/SO(3)$ .

The inverse matrix of  $\Phi$  is explicitly given by

$$\Phi^{-1} = \begin{pmatrix} f + f^{IJ}\omega_I\omega_J & f^{\phi J}\omega_J & f^{\psi J}\omega_J \\ f^{\phi J}\omega_J & f^{\phi\phi} & f^{\phi\psi} \\ f^{\psi J}\omega_J & f^{\phi\psi} & f^{\psi\psi} \end{pmatrix}, \quad (15)$$

and transforms as a second rank contravariant tensor field on the base space.

The current matrix defined by

$$J_i = \Phi^{-1} \partial_i \Phi \quad (16)$$

linearly transforms according to the adjoint representation of  $SL(3, \mathbf{R})$ . This current is conserved, namely every element of  $D_i J^i$  independently vanishes due to the field equations (3) and (4).

The action (12) can be expressed in terms of  $J_i$  or  $\Phi$  as

$$S = \frac{1}{4} \int d\rho dz \rho \text{tr}(J_i J^i), \quad (17)$$

$$= \frac{1}{4} \int d\rho dz \rho \text{tr}(\Phi^{-1} \partial_i \Phi \Phi^{-1} \partial^i \Phi). \quad (18)$$

This action takes a nonlinear  $\sigma$ -model form.

### III. MAZUR IDENTITY

Let us consider two different sets of the field configurations  $\Phi_{[0]}$  and  $\Phi_{[1]}$  satisfying the field equations (3) and (4). To show the coincidence of the two solutions, we will derive the Mazur identity for the nonlinear  $\sigma$ -model on the symmetric space  $SL(3, \mathbf{R})/SO(3)$

A bull's eye  $\odot$  denotes the difference between the value of functional obtained from the field configuration  $\Phi_{[1]}$  and value obtained from  $\Phi_{[0]}$ , *e.g.*,

$$\overset{\odot}{J}^i = J_{[1]}^i - J_{[0]}^i = \Phi_{[1]}^{-1} \partial^i \Phi_{[1]} - \Phi_{[0]}^{-1} \partial^i \Phi_{[0]}. \quad (19)$$

The deviation matrix  $\Psi$  is defined by

$$\Psi = \overset{\odot}{\Phi} \Phi_{[0]}^{-1} = \Phi_{[1]} \Phi_{[0]}^{-1} - \mathbf{1}, \quad (20)$$

where  $\mathbf{1}$  is the unit matrix. The deviation  $\Psi$  vanishes if and only if the two sets of field configurations ( $[1]$  and  $[0]$ ) coincide with each other. Differentiating  $\Psi$ ,

$$D^i \Psi = \Phi_{[1]} \overset{\odot}{J}^i \Phi_{[0]}^{-1}, \quad (21)$$

and taking divergence, we obtain

$$D_i(D^i \Psi) = \Phi_{[1]} D_i \overset{\odot}{J}^i \Phi_{[0]}^{-1} + \Phi_{[1]} \left\{ J_{[1]i} J_{[1]}^i - 2J_{[1]i} J_{[0]}^i + J_{[0]i} J_{[0]}^i \right\} \Phi_{[0]}^{-1}. \quad (22)$$

Due to the current conservation  $D_i J^i = 0$ , the first term of the right hand side of Eq. (22) vanishes. Since  ${}^t J^i = \Phi J^i \Phi^{-1}$ , the second term of the right hand side can be rewritten as

$$\Phi_{[1]} \left\{ J_{[1]i} J_{[1]}^i - 2J_{[1]i} J_{[0]}^i + J_{[0]i} J_{[0]}^i \right\} \Phi_{[0]}^{-1} = \Phi_{[1]} \left( J_{[1]}^i \overset{\odot}{J}_i - \overset{\odot}{J}_i J_{[0]}^i \right) \Phi_{[0]}^{-1} \quad (23)$$

$$= {}^t J_{[1]}^i \Phi_{[1]} \overset{\odot}{J}_i \Phi_{[0]}^{-1} - \Phi_{[1]} \overset{\odot}{J}_i \Phi_{[0]}^{-1} {}^t J_{[0]}^i. \quad (24)$$

Then taking trace, we obtain the identity

$$(D_i D^i \text{tr} \Psi) = \text{tr} \left\{ {}^t \overset{\odot}{J}^i \Phi_{[1]} \overset{\odot}{J}_i \Phi_{[0]}^{-1} \right\}. \quad (25)$$

Since  $D$  is covariant derivative with respect to the abstract flat three-metric (11) and all quantities are independent on  $\varphi$ , the above identity (25) is

$$\partial_a(\rho\partial^a\text{tr}\Psi) = \rho h_{ab}\text{tr}\left\{\overset{\circ}{J}^a\Phi_{[1]}\overset{\circ}{J}^b\Phi_{[0]}^{-1}\right\}, \quad (26)$$

where  $h_{ab}$  is the flat two-dimensional metric

$$h = d\rho^2 + dz^2. \quad (27)$$

Integrating Eq. (26) over the relevant region  $\Sigma = \{(\rho, z)|\rho \geq 0\}$  in  $\rho$ - $z$  plane, and using Green's theorem, we find

$$\oint_{\partial\Sigma} \rho\partial^a\text{tr}\Psi dS_a = \int_{\Sigma} \rho h_{ab}\text{tr}\left\{{}^tJ^a\Phi_{[1]}\overset{\circ}{J}^b\Phi_{[0]}^{-1}\right\} d\rho dz, \quad (28)$$

where the boundary  $\partial\Sigma$  is corresponding to the horizon, the two planes of rotation and infinity. Since the matrix  $\Phi$  has the square root matrix  $g$  as Eq. (14), the integrand of the right hand side of Eq. (28) is written by

$$\rho h_{ab}\text{tr}\left\{{}^tJ^a\Phi_{[1]}\overset{\circ}{J}^b\Phi_{[0]}^{-1}\right\} = \rho h_{ab}\text{tr}\left\{g_{[0]}^{-1}{}^tJ^a g_{[1]}{}^tJ^b g_{[0]}^{-1}\right\} \quad (29)$$

Thus, we obtain the Mazur identity

$$\oint_{\partial\Sigma} \rho\partial^a\text{tr}\Psi dS_a = \int_{\Sigma} \rho h_{ab}\text{tr}\left\{\mathcal{M}^a{}^t\mathcal{M}^b\right\} d\rho dz, \quad (30)$$

where the matrix  $\mathcal{M}$  is defined by

$$\mathcal{M}^a = g_{[0]}^{-1}{}^tJ^a g_{[1]}. \quad (31)$$

When the current difference  $\overset{\circ}{J}^a$  is not zero, the right hand side of the identity (30) is positive. Hence we must have  $\overset{\circ}{J}^a = 0$  if the boundary conditions under which the left hand side of Eq. (30) vanishes are imposed at  $\partial\Sigma$ . Then the difference  $\Psi$  is a constant matrix over the region  $\Sigma$ . The limiting value of  $\Psi$  is zero on at least one part of the boundary  $\partial\Sigma$  is sufficient to obtain the coincidence of two solutions  $\Phi_{[0]}$  and  $\Phi_{[1]}$ .

#### IV. BOUNDARY CONDITIONS AND COINCIDENCE OF SOLUTIONS

When one use the Mazur identity, the boundary conditions for the fields  $\Phi$  (*i.e.*,  $f_{IJ}$  and  $\omega_I$ ) are needed at the infinity, the two planes of rotation and the horizon. We will



require asymptotic flatness, regularity at the two planes of rotation, and regularity at the spherical horizon. Under these conditions, the Mazur identity shows that the coincidence of the solutions.

An asymptotically flat space-time with mass  $M = 3\pi m/8G$ , angular momenta  $J_\phi = \pi ma/4G$  and  $J_\psi = \pi mb/4G$  (where we restrict ourselves to the case in which  $m > a^2 + b^2 + 2|ab|$ ) has metric as the following form:

$$\begin{aligned}
g = & - \left[ 1 - \frac{m}{r^2} + O(r^{-3}) \right] dt^2 - \left[ \frac{2ma}{r^4} + O(r^{-5}) \right] dt(ydx - xdy) \\
& - \left[ \frac{2mb}{r^4} + O(r^{-5}) \right] dt(wdz - zdw) \\
& + \left[ 1 + \frac{m}{2r^2} + O(r^{-3}) \right] [dx^2 + dy^2 + dz^2 + dw^2].
\end{aligned} \tag{32}$$

Here introducing the coordinates

$$x = \sqrt{r^2 + a^2} \sin \theta \cos[\bar{\phi} - \tan^{-1}(a/r)], \tag{33}$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin[\bar{\phi} - \tan^{-1}(a/r)], \tag{34}$$

$$z = \sqrt{r^2 + b^2} \cos \theta \cos[\bar{\psi} - \tan^{-1}(b/r)], \tag{35}$$

$$w = \sqrt{r^2 + b^2} \cos \theta \sin[\bar{\psi} - \tan^{-1}(b/r)], \tag{36}$$

and proceeding further coordinate transformations

$$d\bar{\phi} = d\phi - \frac{a}{r^2 + a^2} dr, \tag{37}$$

$$d\bar{\psi} = d\psi - \frac{b}{r^2 + b^2} dr, \tag{38}$$

then one obtains

$$\begin{aligned}
g = & - \left[ 1 - \frac{m}{r^2} + O(r^{-3}) \right] dt^2 + \left[ \frac{2ma(r^2 + a^2)}{r^4} \sin^2 \theta + O(r^{-3}) \right] dt d\phi \\
& + \left[ \frac{2mb(r^2 + b^2)}{r^4} \cos^2 \theta + O(r^{-3}) \right] dt d\psi \\
& + \left[ 1 + \frac{m}{2r^2} + O(r^{-3}) \right] \times \left[ \frac{r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta}{(r^2 + a^2)(r^2 + b^2)} r^2 dr^2 \right. \\
& \left. + (r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + (r^2 + b^2) \cos^2 \theta d\psi^2 \right].
\end{aligned} \tag{39}$$

Here the metric (39) admits two orthogonal planes of rotation  $\theta = \pi/2$  and  $\theta = 0$ , which are specified by the azimuthal angles  $\phi$  and  $\psi$ , respectively. The planes  $\theta = 0$  and  $\theta = \pi/2$  are

invariant under the rotation with respect to the Killing vector fields  $\partial_\phi$  and  $\partial_\psi$ , respectively. Both angles  $\phi$  and  $\psi$  have period  $2\pi$ . Comparing the asymptotic form (39) with the Weyl-Papapetrou form (6), we derive boundary conditions.

The regularity on invariant planes requires

$$g_{\phi\phi} = f_{\phi\phi} = \sin^2 \theta \tilde{f}_{\phi\phi}, \quad (40)$$

$$g_{\psi\psi} = f_{\psi\psi} = \cos^2 \theta \tilde{f}_{\psi\psi}, \quad (41)$$

$$g_{\phi\psi} = f_{\phi\psi} = \sin^2 \theta \cos^2 \theta \tilde{f}_{\phi\psi}, \quad (42)$$

where the quantities with tilde are regular at both the invariant planes and the black hole horizon.

The asymptotic behavior of  $\tilde{f}_{\phi\phi}$  and  $\tilde{f}_{\psi\psi}$  are derived from Eq. (39), and  $\tilde{f}_{\phi\psi}$  is at most  $O(r^{-1})$  since Killing vectors  $\partial_\phi$  and  $\partial_\psi$  are asymptotically orthogonal.

$$\tilde{f}_{\phi\phi} = r^2 + a^2 + \frac{m}{2} + O(r^{-1}), \quad (43)$$

$$\tilde{f}_{\psi\psi} = r^2 + b^2 + \frac{m}{2} + O(r^{-1}), \quad (44)$$

$$\tilde{f}_{\phi\psi} = O(r^{-1}). \quad (45)$$

Since  $f_{\phi\psi}$  is negligible as compared with  $f_{\phi\phi}$  and  $f_{\psi\psi}$  in the asymptotic region, the leading terms of  $g_{t\phi}$  and  $g_{t\psi}$  are  $f_{\phi\phi}w^\phi$  and  $f_{\psi\psi}w^\psi$ , respectively. Then, we have

$$f_{\phi\phi}w^\phi = \frac{ma \sin^2 \theta}{r^2} + O(r^{-3}), \quad (46)$$

$$f_{\psi\psi}w^\psi = \frac{mb \cos^2 \theta}{r^2} + O(r^{-3}). \quad (47)$$

Thus we obtain

$$w^\phi = \frac{ma}{r^4} + O(r^{-5}), \quad (48)$$

$$w^\psi = \frac{mb}{r^4} + O(r^{-5}). \quad (49)$$

Similarly, we have

$$g_{tt} = -f^{-1}\rho^2 + f_{\phi\phi}w^\phi w^\phi + 2f_{\phi\psi}w^\phi w^\psi + f_{\psi\psi}w^\psi w^\psi \quad (50)$$

$$= -1 + \frac{m}{r^2} + O(r^{-3}). \quad (51)$$

Here  $O(r^{-2})$  term must come from  $-f^{-1}\rho^2$  term since the  $w^I$  are  $O(r^{-4})$ . Therefore  $\rho$  behaves as

$$\rho^2 = [r^4 + (a^2 + b^2)r^2 + O(r)] \sin^2 \theta \cos^2 \theta. \quad (52)$$

$\rho^2$  does not only vanish at  $\phi$ -invariant plane ( $\sin \theta = 0$ ) and  $\psi$ -invariant plane ( $\cos \theta = 0$ ), but also vanishes at the horizon due to the form of the metric (6). Since the horizon has topology of  $S^3$ , let us introduce the spheroidal coordinates on  $\Sigma$  as

$$z = \lambda\mu, \quad (53)$$

$$\rho^2 = (\lambda^2 - c^2)(1 - \mu^2), \quad (54)$$

where  $\mu = \cos 2\theta$ . Then the relevant region is  $\Sigma = \{(\lambda, \mu) | \lambda \geq c, -1 \leq \mu \leq 1\}$ . The boundaries  $\lambda = c$ ,  $\lambda = +\infty$ ,  $\mu = 1$  and  $\mu = -1$  correspond to the horizon, the infinity, the  $\phi$ -invariant plane and the  $\psi$ -invariant plane, respectively. In these coordinates, the two-dimensional metric on  $\Sigma$  is given by

$$h = d\rho^2 + dz^2 = (\lambda^2 - c^2\mu^2) \left( \frac{d\lambda^2}{\lambda^2 - c^2} + \frac{d\mu^2}{1 - \mu^2} \right). \quad (55)$$

The boundary integral in the left hand side of the Mazur identity (30) is explicitly written as

$$\oint_{\partial\Sigma} \rho \partial^a \text{tr} \Psi dS_a = \int_c^\infty d\lambda \left( \sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}} \rho \frac{\partial \text{tr} \Psi}{\partial \mu} \right) \Big|_{\mu=-1} + \int_{-1}^{+1} d\mu \left( \sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}} \rho \frac{\partial \text{tr} \Psi}{\partial \lambda} \right) \Big|_{\lambda=\infty} \\ + \int_\infty^c d\lambda \left( \sqrt{\frac{h_{\lambda\lambda}}{h_{\mu\mu}}} \rho \frac{\partial \text{tr} \Psi}{\partial \mu} \right) \Big|_{\mu=+1} + \int_{+1}^{-1} d\mu \left( \sqrt{\frac{h_{\mu\mu}}{h_{\lambda\lambda}}} \rho \frac{\partial \text{tr} \Psi}{\partial \lambda} \right) \Big|_{\lambda=c}, \quad (56)$$

where

$$\frac{\partial \text{tr} \Psi}{\partial x^a} = \frac{\partial}{\partial x^a} \left[ f_{[1]}^{-1} \left( -\overset{\circ}{f} + f_{[0]}^{IJ} \overset{\circ}{\omega}_I \overset{\circ}{\omega}_J \right) + f_{[0]}^{IJ} \overset{\circ}{f}_{IJ} \right], \quad \text{for } x^a = \lambda, \mu. \quad (57)$$

Here the relation between  $\lambda$  and  $r$  is given by

$$\lambda = \frac{r^2}{2} + \frac{a^2 + b^2}{4} + O(r^{-1}), \quad (58)$$

or

$$r = \sqrt{2}\lambda^{1/2} \left[ 1 - \frac{a^2 + b^2}{8\lambda} + O(\lambda^{-3/2}) \right]. \quad (59)$$

The boundary conditions for  $f_{IJ}$  are summarized as follows:

	$\phi$ -invariant plane	$\psi$ -invariant plane	horizon	infinity
	$\mu \rightarrow +1$	$\mu \rightarrow -1$	$\lambda \rightarrow c$	$\lambda \rightarrow +\infty$
$\tilde{f}_{\phi\phi}$	$O(1)$	$O(1)$	$O(1)$	$2\lambda + (a^2 - b^2 + m)/2 + O(\lambda^{-1/2})$
$\tilde{f}_{\phi\psi}$	$O(1)$	$O(1)$	$O(1)$	$O(\lambda^{-1/2})$
$\tilde{f}_{\psi\psi}$	$O(1)$	$O(1)$	$O(1)$	$2\lambda + (b^2 - a^2 + m)/2 + O(\lambda^{-1/2})$

where

$$f_{\phi\phi} = \frac{(1-\mu)}{2} \tilde{f}_{\phi\phi}, \quad (60)$$

$$f_{\phi\psi} = \frac{(1-\mu)(1+\mu)}{4} \tilde{f}_{\phi\psi}, \quad (61)$$

$$f_{\psi\psi} = \frac{(1+\mu)}{2} \tilde{f}_{\psi\psi}. \quad (62)$$

Next, let us derive the boundary conditions for the twist potentials. By the definition of twist potentials, Eq. (2),

$$\frac{\partial\omega_\phi}{\partial\lambda} = -\frac{f f_{\phi J}}{\lambda^2 - c^2} \frac{\partial w^J}{\partial\mu}, \quad \frac{\partial\omega_\phi}{\partial\mu} = \frac{f f_{\phi J}}{1-\mu^2} \frac{\partial w^J}{\partial\lambda}, \quad (63)$$

$$\frac{\partial\omega_\psi}{\partial\lambda} = -\frac{f f_{\psi J}}{\lambda^2 - c^2} \frac{\partial w^J}{\partial\mu}, \quad \frac{\partial\omega_\psi}{\partial\mu} = \frac{f f_{\psi J}}{1-\mu^2} \frac{\partial w^J}{\partial\lambda}. \quad (64)$$

From the  $\mu$  dependence of  $f_{IJ}$ , the  $\mu$  dependence of the derivatives of the twist potentials are given as follows:

$$\frac{\partial\omega_\phi}{\partial\lambda} = \frac{\partial\omega_\phi}{\partial\mu} = \frac{\partial\omega_\psi}{\partial\lambda} = 0 \quad \text{at } \mu = +1, \quad \frac{\partial\omega_\psi}{\partial\mu} \text{ does not have } (1-\mu) \text{ as a factor}, \quad (65)$$

$$\frac{\partial\omega_\psi}{\partial\lambda} = \frac{\partial\omega_\psi}{\partial\mu} = \frac{\partial\omega_\phi}{\partial\lambda} = 0 \quad \text{at } \mu = -1, \quad \frac{\partial\omega_\phi}{\partial\mu} \text{ does not have } (1+\mu) \text{ as a factor}. \quad (66)$$

In the asymptotic region ( $\lambda \rightarrow +\infty$ ), the derivatives of the twist potentials behave as

$$\frac{\partial\omega_\phi}{\partial\lambda} = O(\lambda^{-3/2}), \quad (67)$$

$$\frac{\partial\omega_\phi}{\partial\mu} = -\frac{ma}{2}(1-\mu) + O(\lambda^{-1/2}). \quad (68)$$

Thus we obtain

$$\omega_\phi = -\frac{ma}{4}\mu(2-\mu) + (1-\mu)^2(1+\mu)O(\lambda^{-1/2}), \quad (69)$$

and similarly

$$\omega_\psi = -\frac{mb}{4}\mu(2+\mu) + (1-\mu)(1+\mu)^2O(\lambda^{-1/2}). \quad (70)$$

Then, of course, the condition that  $\omega_I$  are regular on the horizon is required.

The boundary conditions for  $\omega_I$  are summarized as follows:

	$\phi$ -invariant plane	$\psi$ -invariant plane	horizon	infinity
	$\mu \rightarrow +1$	$\mu \rightarrow -1$	$\lambda \rightarrow c$	$\lambda \rightarrow +\infty$
$\tilde{\omega}_\phi$	$O((1-\mu)^2)$	$O(1+\mu)$	$O(1)$	$O(\lambda^{-1/2})$
$\tilde{\omega}_\psi$	$O(1-\mu)$	$O((1+\mu)^2)$	$O(1)$	$O(\lambda^{-1/2})$

where

$$\omega_\phi = -\frac{ma}{4}\mu(2-\mu) + \tilde{\omega}_\phi, \quad (71)$$

$$\omega_\psi = -\frac{mb}{4}\mu(2+\mu) + \tilde{\omega}_\psi. \quad (72)$$

The behavior of the following quantities which appear in the boundary integral (56) are easily calculated as follows.

	$\phi$ -invariant plane	$\psi$ -invariant plane	horizon	infinity
	$\mu \rightarrow +1$	$\mu \rightarrow -1$	$\lambda \rightarrow c$	$\lambda \rightarrow +\infty$
$\partial \text{tr} \Psi / \partial \lambda$	—	—	$O(1)$	$O(\lambda^{-5/2})$
$\partial \text{tr} \Psi / \partial \mu$	$O(1)$	$O(1)$	—	—
$\rho$	$O(\sqrt{1-\mu})$	$O(\sqrt{1+\mu})$	$O(\sqrt{\lambda-c})$	$O(\lambda)$
$\sqrt{h_{\mu\mu}/h_{\lambda\lambda}}$	—	—	$O(\sqrt{\lambda-c})$	$O(\lambda)$
$\sqrt{h_{\lambda\lambda}/h_{\mu\mu}}$	$O(\sqrt{1-\mu})$	$O(\sqrt{1+\mu})$	—	—

Then, the boundary integral (56) vanishes. The difference matrix  $\Psi$  is constant and has asymptotic behavior as

$$\Psi \rightarrow \begin{pmatrix} O(\lambda^{-3/2}) & O(\lambda^{-7/2}) & O(\lambda^{-7/2}) \\ O(\lambda^{-1/2}) & O(\lambda^{-3/2}) & O(\lambda^{-3/2}) \\ O(\lambda^{-1/2}) & O(\lambda^{-3/2}) & O(\lambda^{-3/2}) \end{pmatrix}, \quad (\lambda \rightarrow +\infty). \quad (73)$$

$\Psi$  vanishes at the infinity, and then  $\Psi$  is zero over  $\Sigma$ . Thus, the two configurations  $\Phi_{[0]}$  and  $\Phi_{[1]}$  coincide with each other.

## V. SUMMARY AND DISCUSSION

We show uniqueness of the asymptotically flat, black hole solution to the five-dimensional vacuum Einstein equation with the regular event horizon homeomorphic to  $S^3$ , admitting two commuting spacelike Killing vector fields and stationary Killing vector field. The solution of this system is determined by only three asymptotic charges, the mass  $M = 3\pi m/8G$  and the two angular momenta  $J_\phi = \pi ma/4G$  and  $J_\psi = \pi mb/4G$ . The five-dimensional Myers-Perry black hole solution is unique in this class.

The vacuum black ring solution fulfills above conditions other than that on the topology of the horizon. There exist two black ring solutions which have same mass and angular

momentum, which means uniqueness property fails for the  $S^2 \times S^1$  event horizon. It is intriguing to investigate how this nonuniqueness occurs.

It will be impossible to extend our argument using the Mazur identity to the six or higher dimensional Myers-Perry black hole solutions. An  $n$ -dimensional space-time admitting  $(n - 3)$  commuting Killing vector fields is always described by nonlinear  $\sigma$ -model as shown by Maison [24]. To derive the Mazur identity for this nonlinear  $\sigma$ -model, all the  $(n - 3)$  Killing vector fields have to be spacelike. However, the  $n$ -dimensional Myers-Perry black hole space-time has only  $[(n - 1)/2]$  commuting spacelike Killing vector fields. Thus our method cannot be used except for the five-dimensional Myers-Perry black hole.

The rigidity theorem in four dimensions claims that the asymptotically flat, stationary analytic space-time is also axi-symmetric [27]. However the existence of additional space-time Killing vector fields is not justified in the case of five-dimensional black holes any longer. Therefore uniqueness shown in the present work does not exclude the possibility of existence of the black hole solutions with less symmetry as suggested by Reall [9].

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